MATH 320 NOTES, WEEKS 5 AND 6

Recall that:

If V is a vector space with dimension n and $\beta \subset V$ has size n. Then TFAE:

(1) β is a basis;
(2) β is linearly independent;
(3) $span(\beta) = V$.

That means that when you prove that a set is a basis, after checking that it has the right number of vectors, it is enough to prove that it is linearly independent, or to prove that it generates the vector space. (You don't have to do both – in the finite dimensional case.)

Also, if $\dim(V) = n$ and $S \subset V$, then

- if S is linearly independent, then $|S| \leq n$, and S can be extended to a basis.
- if S spans V, then $|S| \geq n$ and S contains a basis.

Theorem 1. *Suppose that V is a finite dimensional vector space and W is a subspace. Then $\dim(W) \leq \dim(V)$, and*

$$\dim(W) = \dim(V) \text{ iff } W = V.$$

Proof. Let α be a basis for W. Then α is a linearly independent subset of V and so can be extended to a basis for V. It follows that $\dim(W) = |\alpha| \leq |\beta| = \dim(V)$.

Next we show the "iff" statement. Clearly, if $V = W$, then they have the same dimension.

For the other direction, suppose that $\dim(V) = \dim(W) = n$. We have to show that $V = W$.

As above let α be a basis for W and extend α to a basis β for V. I.e. $\alpha \subset \beta$. Since both V and W have dimension n, we have that $|\alpha| = |\beta| = n$. But then $\alpha = \beta$. So, $V = span(\beta) = span(\alpha) = W$.

□

Example: In F^4, compute the dimensions of the following subspaces.

(1) $W_1 = \{(a_1, a_2, a_3, a_4) \in F^4 \mid a_1 + a_2 = a_3\}$;
(2) $W_2 = \{(a_1, a_2, a_3, a_4) \in F^4 \mid a_1 + a_2 = 0, a_3 + a_4 = 0\}$;
(3) $W_3 = \{(a_1, a_2, a_3, a_4) \in F^4 \mid a_1 = a_2 = a_4 = 0\}$.

Solution:

Note that if a vector $v \in W_1$, it has the form

$$v = (a, b, a + b, c).$$

Then a basis for W_1 is $\beta_1 = \{(1,0,1,0),(0,1,1,0),(0,0,0,1)\}$, and $\dim(W_1) = 3$.

Next, if a vector $v \in W_2$, it has the form

$$v = (a, -a, b, -b).$$

Then a basis for W_2 is $\beta_1 = \{(1,-1,0,0),(0,0,1,-1\}$, and $\dim(W_1) = 2$.

Finally, if a vector $v \in W_3$, it has the form

$$v = (0,0,a,0).$$

Then a basis for W_3 is $\beta_1 = \{(0,0,1,0\}$, and $\dim(W_1) = 1$.

Example: In $M_{3,3}(F)$, compute the dimensions of the following subspaces.

(1) $W_1 = \{A \mid tr(A) = 0\}$;
(2) $W_2 = \{A \mid A \text{ is symmetric}\}$;
(3) $W_1 \cap W_2$.
(4) $W_1 + W_2$

Solution:

If $A = (a_{ij}) \in W_1$, then $tr(A) = a_{11} + a_{22} + a_{33} = 0$. Note that if $i \neq j$, then $tr(E_{ij}) = 0$, and $tr(E_{ii}) = 1$. A basis for W_1 is

$$\beta = \{E_{ij} \mid i \neq j\} \cup \{E_{11} - E_{33}, E_{22} - E_{33}\}.$$

Then $\dim(W_1) = 8$.

If $A = (a_{ij}) \in W_2$, then A is symmetric and so $a_{ij} = a_{ji}$ for each $1 \leq i, j \leq 3$. Note that for all $1 \leq i \leq 3$, E_{ii}, is symmetric. A basis for W_2 is

$$\gamma = \{E_{11}, E_{22}, E_{33}\} \cup \{E_{ij} + E_{ji} \mid i \neq j\}$$

Then $\dim(W_2) = 6$.

If $A = (a_{ij}) \in W_1 \cap W_2$, then both $a_{ij} = a_{ji}$ for each $1 \leq i, j \leq 3$ and $a_{11} + a_{22} + a_{33} = 0$ must be true. A basis is

$$\alpha = \{E_{11} - E_{33}, E_{22} - E_{33}\} \cup \{E_{ij} + E_{ji} \mid i \neq j\}.$$

Then $\dim(W_1 \cap W_2) = 5$.

$W_1 + W_2 = \{A + B \mid tr(A) = 0 \text{ and } B \text{ is symmetric}\}$. If $i \neq j$, then $E_{ij} \in W_1 \subset W_1 + W_2$. If $i \leq 3$, $E_{ii} \in W_2 \subset W_1 + W_2$. So the standard basis $\{E_{ij} \mid 1 \leq i, j \leq 3\} \subset W_1 + W_2$, so $V = W_1 + W_2$, and $\dim(W_1 + W_2) = \dim(V) = 9$.

Note that

$$\dim(W_1 + W_2) = 9 = 8 + 6 - 5 = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

2.1 Linear transformations.

Definition 2. *Let V and W be vector spaces over a filed F. A map $L : V \to W$ is called* **a linear transformation** *if for all $x, y \in V$ and $c \in F$,*

(1) *$L(x+y) = L(x) + L(y)$, and*
(2) *$L(cx) = cL(x)$*

If L is as above, the range of L, $\mathrm{ran}(L) = \{L(x) \mid x \in V\}$ *and the kernel of the nullspace of L is* $\ker(L) = \{x \in V \mid L(x) = \vec{0}\}$

Examples of linear transformations

(1) $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L((a,b)) = (a,0)$.
(2) $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L((a,b,c)) = (a+b, c, 2a)$.
(3) The trace $tr : M_{n,n}(F) \to F$.
(4) The derivative $D : P(F) \to P(F)$ given by $D(p) = p'$.
(5) Matrix multiplication: Let $A \in M_{n,m}$ $L_A : F^m \to F^n$ given by

$$L_A(x) = Ax.$$

And here are some maps that are NOT linear transformations:

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f((a,b)) = (a+b, a^2)$.
- $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f((a,b)) = (a,1)$.

Lemma 3. *Let $L : V \to W$ be a linear transformation, where V, W are vector spaces over F. Then:*

- *$L(\vec{0}_V) = \vec{0}_W$*
- *$L(-x) = -L(x)$ for all x,*
- *$L(x-y) = L(x) - L(y)$ and $L(cx+y) = cL(x) + L(y)$.*

More generally, we can use linearity on any linear combination and get:

$$L(a_1x_1 + ... + a_nx_n) = a_1L(x_1) + ... + a_nL(x_n).$$

Proof. For the first item, let x be an arbitrary vector in V. Then $L(\vec{0}) = L(0x) = 0L(x) = \vec{0}$.

For the second item, $L(-x) = L(-1x) = -1L(x) = L(x)$. The rest are left as exercises.

□

Lemma 4. *Let $L : V \to W$ be a linear transformation, where V, W are vector spaces over F. Then:*

- $\ker(L)$ *is a subspace of V, and*
- $\mathrm{ran}(L)$ *is a subspace of W.*

Proof. Suppose that x, y are both in $\ker(L)$ and c is a scalar. We have to show that $cx + y \in \ker(L)$. To do that, we compute $L(cx+y) = cL(x) + L(y) = c\vec{0} + \vec{0} = \vec{0}$. It follows that $\ker(L)$ is a subspace.

For the range, suppose that y_1, y_2 are in $\mathrm{ran}(L)$ and c is a scalar. We have to show that $cx + y \in \mathrm{ran}(L)$. First, by definition of the range, there are some vectors x_1, x_2 in V, such that $L(x_1) = y_1$ and $L(x_2) = y_2$. Then $L(cx_1 + x_2) = cL(x_1) + L(x_2) = cy_1 + y_2 \in range(L)$. □

Definition 5. *Suppose that $L : V \to W$ is a linear transformation. L is* **one-to-one** *if whenever x, y are such that $L(x) = L(y)$, then $x = y$. L is* **onto** *if* $\mathrm{ran}(L) = W$.

Lemma 6. *Suppose that $L : V \to W$ is a linear transformation. Then L is one-to-one iff* $\ker L = \{\vec{0}\}$.

Proof. For the easier direction,suppose that L is one-to-one. Let $x \in \ker(L)$. Then $L(x) = \vec{0} = L(\vec{0})$. Since L is one-to-one , it follows that $x = \vec{0}$.

For the other direction, suppose that $\ker L = \{\vec{0}\}$. We have to show that L is one-to-one. Suppose that $L(x) = L(y)$. Then $\vec{0} = L(x) - L(y) = L(x - y)$, so $x - y \in \ker(L)$. It follows that $x - y = \vec{0}$, and $x = y$.

□

Lemma 7. *Suppose that $L : V \to W$ is a linear transformation, and $\{x_1, ..., x_n\}$ are vectors in V. Then,*

1. *If $\{L(x_1), ..., L(x_n)\}$ is linearly independent, then $\{x_1, ..., x_n\}$ are linearly independent.*
2. *If span$(\{x_1, ..., x_n\}) = V$, then span$(\{L(x_1), ..., L(x_n)\}) = \mathrm{ran}(L)$.*

Proof. For item (1), suppose that $a_1x_1 + ...a_nx_n = \vec{0}$. Then, applying L to both sides, we have that

$$L(a_1x_1 + ...a_nx_n) = a_1L(x_1) + ...a_nL(x_n) = \vec{0}.$$

Since $\{L(x_1), ..., L(x_n)\}$ is linearly independent, it follows that $a_1 = a_2 = ... = a_n = 0$.

For item (2), suppose that *span*$(\{x_1, ..., x_n\}) = V$. Let $z \in \mathrm{ran}(L)$. Then for some $x \in V$, $z = L(x)$. Since $\{x_1, ..., x_n\}$ spans V, let $a_1, ..., a_n$ be scalars such that $x = a_1x_1 + ... + a_nx_n$. Then $z = L(x) = L(a_1x_1 + ... + a_nx_n) = a_1L(x_1) + ... + a_nL(x_n)$.

So, $z \in$ *span*$(\{L(x_1), ..., L(x_n)\})$. Since z was arbitrary, it follows that *span*$(\{L(x_1), ..., L(x_n)\}) = \mathrm{ran}(L)$. □